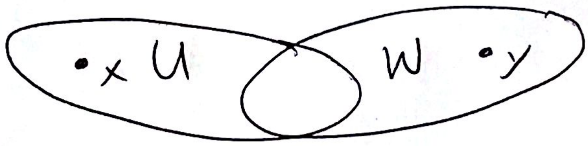
Midterm I Solutions

1) Ex: $U = \{(x,0,0) \mid x \in \mathbb{R}\}$, $W = \{(0,y,0) \mid y \in \mathbb{R}\}$ are both subspaces of $\mathbb{R}^3$ but $U \cup W$ is not a subspace. To see this observe that $(1,0,0) \in U \cup W$ and $(0,1,0) \in U \cup W$ but $(1,0,0) + (0,1,0) = (1,1,0) \notin U \cup W$.

Assume that $U \cup W$ is a subspace. Proceed by contradiction and suppose that $U \not\subset W$ and $W \not\subset U$. Take two points $x \in U$, $x \notin W$ and $y \in W$, $y \notin U$.



Since $U \cup W$ is a subspace $x+y \in U \cup W$, say $x+y \in U$. Then since U is a subspace $(x+y) - x \in U$ but $(x+y) - x = y \notin U$, a contradiction. We conclude that either $U \subset W$ or $W \subset U$.

2) $$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 5 \\ 1 & 4 & -3 & -3 & 6 \\ 2 & 3 & -1 & 4 & 8 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 5 \\ 0 & 2 & -2 & -4 & 1 \\ 0 & -1 & 1 & 2 & -2 \end{array}\right)$$ 2

$$\to \left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 5 \\ 0 & 1 & -1 & -2 & 2 \\ 0 & 0 & 0 & 0 & -3 \end{array}\right)$$

$\boxed{\text{No Solution}}$

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3) $$c_1\begin{pmatrix}1\\1\\0\end{pmatrix} + c_2\begin{pmatrix}2\\0\\-1\end{pmatrix} + c_3\begin{pmatrix}-3\\1\\1\end{pmatrix} = 0$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right) \Rightarrow c_1 = c_2 = c_3 = 0$$

$\boxed{\text{Linearly Independent}}$

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4) $\text{rank}(T) + \text{nullity}(T) = 4 = \dim \mathbb{R}^4$. 3

Since $\left\{ \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ are LI

get $\text{rank}(T) = 3$ as $R(T) \subset \mathbb{R}^3$.

$$\Rightarrow \boxed{\dim N(T) = 4 - 3 = 1.}$$

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5) $\alpha = \{1, x, x^2\} \to \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

$\beta = \{x^2+x,\ x-1,\ x^2+x+1\} \to \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

$$Q^{-1} = [I]_\beta^\alpha = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Find Q:

$$\left(\begin{array}{ccc|ccc} 0 & -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & 2 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right)$$

$$\Rightarrow \boxed{Q = [I]_\alpha^\beta = \begin{pmatrix} -1 & -1 & 2 \\ 0 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix}.}$$

$$\boxed{[1+x]_\beta = Q \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}}$$

6. Assume LT is onto. $V \xrightarrow{T} W \xrightarrow{L} Z$ $\boxed{4}$

If L is not onto then $\exists$ $z \in Z$ with $L(w) \neq z$ for any ~~z~~ w. In particular $L(T(v)) \neq z$ for any $v \in V$, a contradiction.

T need not be onto. $\boxed{\text{NO!}}$

$T : \mathbb{R} \to \mathbb{R}^2$ by $T(a) = (a, 0)$.

$L : \mathbb{R}^2 \to \mathbb{R}$ by $L(a, b) = a$

Then $LT(a) = a$ is onto but T is not.

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7. $P(x) = (x+i)^2 (x-i)^2 (x+3)(x+1)$

$$\Rightarrow \boxed{y(t) = c_1 e^{-it} + c_2 t e^{-it} + c_3 e^{it} + c_4 t e^{it} + c_5 e^{-3t} + c_6 e^{-t}}.$$